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Secular Perturbations of the Planets.

By G. W. Hill.

Gauss first clearly indicated the rôle elliptic functions play in this subject.* Halphen has since presented the investigation in a very elegant manner.† The modifications made by the latter in the procedure of Gauss are chiefly the transference of the origin of rectangular coordinates to the attracted planet, and, instead of the differential of the eccentric anomaly, the adoption of the element of area described by the radius of the disturbing planet expressed in terms of the differentials of the rectangular coordinates. He also appeals to the qualities of the cone formed by the orbit of the attracting planet as contour of base and the position of the attracted planet as vertex; this improvement, however, had been previously indicated by Bour.‡ A remarkable degree of elegance is attained by these changes; but it seems to me that additional statements are needed to show the connection with the astronomical problem which originally suggested the investigation; for Halphen, like Gauss, treats only the attraction of a certain form of ring. This, of course, is to ignore the second integration which the problem demands. Perhaps, therefore, I shall be pardoned if I here attempt to supply the mentioned lack.

In the fashion of Halphen we take the attracted planet as the origin of rectangular coordinates, but the orientation of the axes is, for the present, left indeterminate. The coordinates of the attracting planet we denote by x, y, z; and the coordinates of the Sun, which are the negatives of those of the attracted planet referred to the Sun, will be x_0, y_0, z_0. Let ρ be the distance of the attracting planet from the origin, so that $\rho^2 = x^2 + y^2 + z^2$; and let g denote the planet's mean anomaly. Then the secular perturbations of the attracted planet depend on the three definite integrals

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* Gauss, Werke, vol. III, pp. 331-355.

† G. H. Halphen, Traité des Fonctions Elliptiques et de leurs Applications, Tom. II, pp. 310-328.

‡ Journal de l'École Polytechnique, Cahier XXXVI, pp. 59-84.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{x}{\rho^3}\,dg,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{y}{\rho^3}\,dg,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{z}{\rho^3}\,dg.$$

But while g is passing from 0 to 2π, the area described by the radius of the planet augments from 0 to πab, if a and b are severally the major and minor semi-axes. Thus, if σ denote this varying area, the preceding integrals may be written

$$\frac{1}{\pi ab}\int_0^{\pi ab}\frac{x}{\rho^3}\,d\sigma,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{y}{\rho^3}\,d\sigma,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{z}{\rho^3}\,d\sigma.$$

The tetrahedron, with $d\sigma$ as base and the origin as vertex, has, for volume one sixth of the following expression:

$$6V = x_0(ydz - zdy) + y_0(zdx - xdz) + z_0(xdy - ydx).$$

But, if h denote the perpendicular from the origin on the plane of the orbit of the attracting planet, we also have $3V = hd\sigma$. Hence

$$d\sigma = \frac{x_0}{2h}(ydz - zdy) + \frac{y_0}{2h}(zdx - xdz) + \frac{z_0}{2h}(xdy - ydx).$$

Then, if we derive the quantities P_x, P_y, P_z, etc., from integrating the expressions

$$dP_x = \tfrac{1}{2}\frac{x(ydz - zdy)}{\rho^3},\quad dP_y = \tfrac{1}{2}\frac{y(ydz - zdy)}{\rho^3},\quad dP_z = \tfrac{1}{2}\frac{z(ydz - zdy)}{\rho^3},$$

$$dQ_x = \tfrac{1}{2}\frac{x(zdx - xdz)}{\rho^3},\quad dQ_y = \tfrac{1}{2}\frac{y(zdx - xdz)}{\rho^3},\quad dQ_z = \tfrac{1}{2}\frac{z(zdx - xdz)}{\rho^3},$$

$$dR_x = \tfrac{1}{2}\frac{x(xdy - ydx)}{\rho^3},\quad dR_y = \tfrac{1}{2}\frac{y(xdy - ydx)}{\rho^3},\quad dR_z = \tfrac{1}{2}\frac{z(xdy - ydx)}{\rho^3},$$

around the whole orbit, the integrals above, which we will denote by X, Y, Z will be given by the following expressions:

$$X = \frac{1}{\pi abh}(x_0P_x + y_0Q_x + z_0R_x),$$

$$Y = \frac{1}{\pi abh}(x_0P_y + y_0Q_y + z_0R_y),$$

$$Z = \frac{1}{\pi abh}(x_0P_z + y_0Q_z + z_0R_z).$$

The nine quantities involved in these expressions and obtained through integration are homogeneous and of the dimension zero with respect to the linear unit.

Moreover, if, in the cone formed by the orbit of the attracting planet as directrix and the origin as vertex, the plane of the base is shifted in any manner whatever, these nine quantities remain unchanged. For example, consider the expressions

$$\frac{x\,(ydz - zdy)}{\rho^3}, \quad \frac{y\,(ydz - zdy)}{\rho^3}, \quad \frac{z\,(ydz - zdy)}{\rho^3}.$$

If we put

$$x = \rho \sin\theta, \quad y = \rho\cos\theta\cos\lambda, \quad z = \rho\cos\theta\sin\lambda,$$

they are transformed into

$$\sin\theta\cos^2\theta\, d\lambda, \quad \cos^3\theta\cos\lambda\, d\lambda, \quad \cos^3\theta\sin\lambda\, d\lambda$$

Let the equation of the cone be

$$Ax^2 + By^2 + Cz^2 + Dyz + Fzx + Fxy = 0,$$

or

$$A\tan^2\theta + B\cos^2\lambda + C\sin^2\lambda + D\sin\lambda\cos\lambda + E\tan\theta\sin\lambda + F\tan\theta\cos\lambda = 0.$$

Since, from this equation, θ is obtainable as a function of λ, it is evident that the integrals of the preceding differential expressions, extended to the whole course of variation of λ, depend solely on the elements of the cone and are altogether independent of the plane section called the base.

A simple addition of the differentials shows that

$$P_x + Q_y + R_z = 0.$$

Also we have

$$d(Q_z - R_y) = \tfrac{1}{2}\, d\left(\frac{x}{\rho}\right), \quad d(R_x - P_z) = \tfrac{1}{2}\, d\left(\frac{y}{\rho}\right), \quad d(P_y - Q_x) = \tfrac{1}{2}\, d\left(\frac{z}{\rho}\right).$$

But the second members of these equations, integrated along the orbit to the point of beginning, give zero as the result: hence

$$Q_z = R_y, \quad R_x = P_z, \; P_y = Q_x.$$

Thus

$$X = \frac{1}{\pi abh}\,(x_0 P_x + y_0 Q_x + z_0 P_z),$$

$$Y = \frac{1}{\pi abh}\,(x_0 Q_x + y_0 Q_y + z_0 R_y),$$

$$Z = \frac{1}{\pi abh}\,(x_0 P_z + y_0 R_y + z_0 R_z).$$

And, if we put

$$\Phi = \frac{1}{2\pi abh}\left[x_0^2 P_x + y_0^2 Q_y + z_0^2 R_z + 2x_0 y_0 Q_x + 2y_0 z_0 R_y + 2z_0 x_0 P_z\right],$$

we shall have

$$X = \frac{\partial\Phi}{\partial x_0}, \quad Y = \frac{\partial\Phi}{\partial y_0}, \quad Z = \frac{\partial\Phi}{\partial z_0}.$$

The orientation of the axes of coordinates which serve to define the variables x, y, z, x_0, y_0, z_0 has been left undetermined; but now suppose that the axes of symmetry of the cone are employed for this purpose. Then the equation of the cone takes the form

$$\frac{x^2}{G_x} + \frac{y^2}{G_y} + \frac{z^2}{G_z} = 0,$$

G_x, G_y, G_z being constants of which two are of one sign and the other of the opposite sign. Plainly, if this equation is satisfied by the set of values x, y, z, it is satisfied by any of the eight sets $\pm x$, $\pm y$, $\pm z$. Consequently, each positive element of the six quantities Q_z, R_y, R_x, P_z, P_y, Q_x is accompanied by a corresponding negative element. Thus, in this case, these quantities vanish. With this selection of axes we, therefore, have

$$X = \frac{x_0}{\pi abh} P_x, \quad Y = \frac{y_0}{\pi abh} Q_y, \quad Z = \frac{z_0}{\pi abh} R_z.$$

The naming of the coordinates is, of course, arbitrary, but, to settle the choice, we suppose that G_x, G_y, G_z are in the order of algebraic magnitude, the first and second being negative, while the last is positive. The equation of the cone appears to involve three variables, but, as we may divide the left member by the square of any one of them, it is, in reality, a relation between two variables; thus, but one variable can be regarded as independent. The equation is then satisfied if we make

$$x = \varepsilon\sqrt{-G_x}\cos T, \quad y = \varepsilon\sqrt{-G_y}\sin T, \quad z = \varepsilon\sqrt{G_z},$$

where ε is an indeterminate which disappears when the substitution is made in dP_x, dQ_y, dR_z, and T is the new variable introduced by Gauss and may be regarded as indicating the position of the planet in its orbit; its function, in this respect, being precisely similar to those fulfilled by the mean, eccentric and true anomalies, and it may thus, with propriety, be designated as a *perspective anomaly*. When T goes from 0 to 2π, the planet makes a complete circuit of its orbit.

The substitution made, we have

$$\begin{aligned} ydz - zdy &= - \varepsilon^2\sqrt{- G_y G_z} \cos TdT, \\ zdx - xdz &= - \varepsilon^2\sqrt{- G_x G_z} \sin TdT, \\ xdy - ydx &= \varepsilon^2 \sqrt{G_x G_y}\, dT, \\ \rho^2 &= \varepsilon^2 [G_z - G_y \sin^2 T - G_x \cos^2 T]. \end{aligned}$$

The quantities G_x, G_y, G_z are usually determined in such a way that $G_x G_y G_z = a^2 b^2 h^2$; also we may introduce k the modulus of the elliptic integrals involved and m such that

$$k^2 = \frac{G_y - G_x}{G_z - G_x}, \quad m = \sqrt{G_z - G_x}.$$

Then our integrals take the forms

$$X = - \frac{x_0}{m^3} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^2 TdT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}}, \quad Y = - \frac{y_0}{m^3} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin^2 TdT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}},$$

$$Z = \frac{z_0}{m^3} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{dT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}}.$$

The methods of evaluating these definite integrals severally proposed by Legendre, Gauss and Jacobi have all about the same degree of rapid convergence but that of the last is to be preferred because it expresses the values explicitly in terms of a parameter q called the *nome*. Putting $k = \sin\theta$, q can be derived from the equation

$$\frac{q + q^9 + q^{25} + \ldots.}{1 + 2(q^4 + q^{16} + q^{36} + \ldots.)} = \left[\frac{\sin\frac{\theta}{2}}{1 + \sqrt{\cos\theta}} \right]^2.$$

The solution is most readily accomplished by the method of tentation. Then, if we adopt two functions of q, K and L such that

$$K = \frac{4}{\cos^2\theta (1 + \sqrt{\cos\theta})^2} [1 + 2(q^4 + q^{16} + \ldots.)]^2,$$

$$L = \frac{(1 + \sqrt{\cos\theta})^3}{\sin^2\theta \cos^{\frac{5}{2}}\theta} \frac{q - 4q^4 + 9q^9 - 16q^{16} + \ldots.}{[1 + 2(q^4 + q^{16} + \ldots.)]^3},$$

we have

$$X = - \frac{x_0}{m^3} L \cos^2\theta, \quad Y = \frac{y_0}{m^3}(L - K), \quad Z = \frac{z_0}{m^3}(K - L \sin^2\theta).*$$

It will be seen from these expressions that, if X, Y, Z are regarded as the components along the axes of coordinates of a force acting on the attracted planet, one elliptic integral suffices for determining the orientation of the resultant, but that an additional one is required if the magnitude of the latter is to be found. It will be an advantage, therefore, if instead of tabulating K and L as functions of k, q or θ, we take two other quantities M and $\varkappa$, such that

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* For the proof of these formulas, reference may be made to Bertrand, Calcul Intégral, Liv. III., Chap. VII.

$$M = K - L\sin^2\theta, \quad \sin^2\varkappa = \frac{L\cos^2\theta}{K - L\sin^2\theta}, \quad \cos^2\varkappa = \frac{K - L}{K - L\sin^2\theta}.$$

Then we shall have

$$X = -\frac{M}{m^3}\sin^2\varkappa . x_0, \quad Y = -\frac{M}{m^3}\cos^2\varkappa . y_0, \quad Z = \frac{M}{m^3}z_0.$$

Let R denote the magnitude of the resultant, H and Λ severally the latitude and longitude of the point in the heavens towards which it is directed; the circles of reference being the principal axes of the sphero-conic traced in the heavens by the frequently mentioned cone. Also let r_0 denote the distance of the Sun from the attracted planet and η_0, λ_0 severally its latitude and longitude referred to the same circles. Then our equations will stand

$$\begin{aligned}
R\cos H\sin\Lambda &= -\frac{M}{m^3}\sin^2\varkappa . r_0\cos\eta_0\sin\lambda_0,\\
R\sin H &= -\frac{M}{m^3}\cos^2\varkappa . r_0\sin\eta_0,\\
R\cos H\cos\Lambda &= \frac{M}{m^3} . r_0\cos\eta_0\cos\lambda_0.
\end{aligned}$$

If we put

$$N = \frac{M}{m^3}r_0\cos\eta_0,$$

we shall have

$$\tan\Lambda = -\sin^2\varkappa\tan\lambda_0, \quad \tan H = -\frac{\cos^2\varkappa\tan\eta_0\cos\Lambda}{\cos\lambda_0}, \quad R = \frac{N\cos\lambda_0}{\cos H\cos\Lambda}.$$

As, except in very particular cases, it is not easy to select at the outset the axes of symmetry of the cone for the adopted axes of coordinates, we must find the position of these axes in reference to another system which is known. The equation of the cone having a very complicated expression when the axes of coordinates are quite general, we select a particular system such that the treatment may be as easy as possible. Let the axis of x have the orientation of the line going from the centre of the ellipse described by the attracting planet to its perihelion, that of y the orientation of the line going from the centre to the point where the eccentric anomaly is 90° and that of z the orientation of the line going from the centre to the north pole of the plane of the orbit. Let the coordinates of the centre of the ellipse be, in their order, A, B, C; we prefer these designations although, with e as the eccentricity and the previous notation, they have the equivalents

$$A = x_0 - ae, \quad B = y_0, \quad C = z_0.$$

Then the equations of the orbit of the attracting planet are

$$\left(\frac{x-A}{a}\right)^2 + \left(\frac{y-B}{b}\right)^2 = 1, \quad z = C.$$

In order to have the equation of the cone so frequently mentioned it is only necessary to multiply the several terms of the first equation by factors selected from the equivalent quantities $1 = \frac{z}{C} = \frac{z^2}{C^2}$, in such a way that they may all become homogeneous and of two dimensions in x, y, z. Thus the equation of the cone may be written in the shape

$$\frac{z^2}{C^2} - \left[\frac{x - \frac{A}{C}z}{a}\right]^2 - \left[\frac{y - \frac{B}{C}z}{b}\right]^2 = 0.$$

This equation is not, in general, referred to the axes of symmetry, hence we proceed to make the linear transformation of variables which will bring this about. Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ denote the rectangular coordinates referred to the symmetric axes of the cone, and write the formulas of transformation thus:

$$\begin{aligned}
\mathrm{X} &= \alpha x + \beta y + \gamma z, & \alpha^2 + \beta^2 + \gamma^2 &= 1, & \alpha\alpha' + \beta\beta' + \gamma\gamma' &= 0, \\
\mathrm{Y} &= \alpha' x + \beta' y + \gamma' z, & \alpha'^2 + \beta'^2 + \gamma'^2 &= 1, & \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' &= 0, \\
\mathrm{Z} &= \alpha'' x + \beta'' y + \gamma'' z, & \alpha''^2 + \beta''^2 + \gamma''^2 &= 1, & \alpha''\alpha + \beta''\beta + \gamma''\gamma &= 0,
\end{aligned}$$

the inverse of which are

$$\begin{aligned}
x &= \alpha\mathrm{X} + \alpha'\mathrm{Y} + \alpha''\mathrm{Z}, & \alpha^2 + \alpha'^2 + \alpha''^2 &= 1, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' &= 0, \\
y &= \beta\mathrm{X} + \beta'\mathrm{Y} + \beta''\mathrm{Z}, & \beta^2 + \beta'^2 + \beta''^2 &= 1, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' &= 0, \\
z &= \gamma\mathrm{X} + \gamma'\mathrm{Y} + \gamma''\mathrm{Z}, & \gamma^2 + \gamma'^2 + \gamma''^2 &= 1, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' &= 0.
\end{aligned}$$

The equation of the cone, after substitution of the new variables, should take the form

$$\frac{\mathrm{X}^2}{G_x} + \frac{\mathrm{Y}^2}{G_y} + \frac{\mathrm{Z}^2}{G_z} = 0,$$

where G_x, G_y, G_z are constants and functions of the quantities A, B, C, a, b. They are the roots of a certain cubic which may be obtained in the following way: Let $V = 0$ denote the first form of the equation of the cone, and set

$$\frac{x}{G} = \tfrac{1}{2}\frac{\partial V}{\partial x}, \quad \frac{y}{G} = \tfrac{1}{2}\frac{\partial V}{\partial y}, \quad \frac{z}{G} = \tfrac{1}{2}\frac{\partial V}{\partial z}.$$

From these equations, which are linear in x, y, z, eliminate these variables; the result is a cubic in G whose roots are the values of G_x, G_y, G_z. In the special form of V with which we have to deal, these equations, after a slight modification, can be thus stated:

$$\frac{x}{G} = \frac{A}{C}\,\frac{z}{G + a^2}, \quad \frac{y}{G} = \frac{B}{C}\,\frac{z}{G + b^2}, \quad \frac{z}{G} = \frac{z}{C^2} - \frac{A}{C}\,\frac{x}{G} - \frac{B}{C}\,\frac{y}{G}.$$

The elimination of x, y, z from these equations gives

$$\frac{A^2}{G + a^2} + \frac{B^2}{G + b^2} + \frac{C^2}{G} = 1.$$

It is known that the roots of this equation in G are all real, two of them being negative and one positive. The mode of assignment of these roots as values of G_x, G_y, G_z has already been described. Thus we are enabled to discover the values of the latter without reference to the nine coefficients α, β, γ, etc. But in our further progress we shall need to know the latter. They are readily found from the first and second equations of the penultimate group combined with the conditions $\alpha^2 + \beta^2 + \gamma^2 = 1$, etc. In the first we make $x = \alpha$, $y = \beta$, $z = \gamma$ and set G_x for G; again make $x = \alpha'$, $y = \beta'$, $z = \gamma'$ and set G_y for G; lastly make $x = \alpha''$, $y = \beta''$, $z = \gamma''$ and set G_z for G. Thus we have the equations

$$\alpha = \frac{A}{C}\,\frac{G_x}{G_x + a^2}\,\gamma, \quad \beta = \frac{B}{C}\,\frac{G_x}{G_x + b^2}\,\gamma,$$

$$\gamma^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_x}{G_x + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_x}{G_x + b^2}\right)^2},$$

$$\alpha' = \frac{A}{C}\,\frac{G_y}{G_y + a^2}\,\gamma', \quad \beta' = \frac{B}{C}\,\frac{G_y}{G_y + b^2}\,\gamma',$$

$$\gamma'^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_y}{G_y + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_y}{G_y + b^2}\right)^2},$$

$$\alpha'' = \frac{A}{C}\,\frac{G_z}{G_z + a^2}\,\gamma'', \quad \beta'' = \frac{B}{C}\,\frac{G_z}{G_z + b^2}\,\gamma'',$$

$$\gamma''^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_z}{G_z + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_z}{G_z + b^2}\right)^2}.$$

These equations furnish the values of the nine coefficients of the transformation. The signs of the three γ's are indeterminate: one may take them as positive.

The cubic in G furnishes us with the relations

$$\frac{A^2}{G_x + a^2} + \frac{B^2}{G_x + b^2} + \frac{C^2}{G_x} = 1,$$
$$\frac{A^2}{G_y + a^2} + \frac{B^2}{G_y + b^2} + \frac{C^2}{G_y} = 1,$$
$$\frac{A^2}{G_z + a^2} + \frac{B^2}{G_z + b^2} + \frac{C^2}{G_z} = 1.$$

Regarding A^2, B^2, C^2 as the unknowns in these equations, their solution gives

$$A^2 = \frac{(G_x + a^2)(G_y + a^2)(G_z + a^2)}{a^2(a^2 - b^2)},$$
$$B^2 = \frac{(G_x + b^2)(G_y + b^2)(G_z + b^2)}{b^2(b^2 - a^2)},$$
$$C^2 = \frac{G_x G_y G_z}{a^2 b^2}.$$

By means of these values we can eliminate A, B, C from the foregoing expressions for the coefficients of transformation, and thus obtain values depending only on the five quantities G_x, G_y, G_z, a, b. We have thus

$$\alpha^2 = \frac{1}{a^2(a^2 - b^2)} \frac{G_x(G_x + b^2)(G_y + a^2)(G_z + a^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\beta^2 = \frac{1}{b^2(b^2 - a^2)} \frac{G_x(G_x + a^2)(G_y + b^2)(G_z + b^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\gamma^2 = \frac{1}{a^2b^2} \frac{G_y G_z (G_x + a^2)(G_x + b^2)}{(G_x - G_y)(G_x - G_z)}.$$

The values of α'^2, β'^2, γ'^2 and again of α''^2, β''^2, γ''^2, are obtained from the preceding by simply making a cyclical permutation of the subscripts attached to the G; firstly from x, y, z into y, z, x, secondly from x, y, z into z, x, y. On account of the divisor $a^2 - b^2$, small when the eccentricity of the planet's orbit is small, some of these formulas labor under a disadvantage.

In computing the values of the definite integrals

$$\frac{1}{\pi ab}\int \frac{x}{\rho^3}\,d\sigma, \quad \frac{1}{\pi ab}\int \frac{y}{\rho^3}\,d\sigma, \quad \frac{1}{\pi ab}\int \frac{z}{\rho^3}\,d\sigma,$$

where x, y, z are referred to the system of axes determined by those of the ellipse described by the attracting planet, we may suppose that the four quantities x_0, y_0, z_0, h are simply constants and unaffected by the transformation we have just made to pass from the system of coordinates x, y, z to that of X, Y, Z. If, for the sake of discrimination, we designate the components along the mentioned axes as X', Y', Z', reserving X, Y, Z for the components along the symmetric axes, we evidently have

$$X' = \alpha X + \alpha' Y + \alpha'' Z, \quad Y' = \beta X + \beta' Y + \beta'' Z, \quad Z' = \gamma X + \gamma' Y + \gamma'' Z.$$

The components of the right members are determined the moment we have solved the cubic in G, for which the coordinates A, B, C furnish the requisite data. We have only to make

$$x_0 = A + ae, \quad y_0 = B, \quad z_0 = C, \quad h = \sqrt{C^2},$$

where h is always to be taken positively. Thus

$$\begin{aligned}
X' &= -\alpha\,(A + ae)\frac{M}{m^3}\sin^2\varkappa - \alpha' B\frac{M}{m^3}\cos^2\varkappa + \alpha'' C\frac{M}{m^3},\\
Y' &= -\beta\,(A + ae)\frac{M}{m^3}\sin^2\varkappa - \beta' B\frac{M}{m^3}\cos^2\varkappa + \beta'' C\frac{M}{m^3},\\
Z' &= -\gamma\,(A + ae)\frac{M}{m^3}\sin^2\varkappa - \gamma' B\frac{M}{m^3}\cos^2\varkappa + \gamma'' C\frac{M}{m^3}.
\end{aligned}$$

Substituting for α, α', α'', β, β', β'' their values in terms of γ, γ', γ'', if we put

$$\begin{aligned}
U_x &= -\frac{A + ae}{m^3}\,\frac{G_x\gamma}{C}\,M\sin^2\varkappa,\\
U_y &= -\frac{B}{m^3}\,\frac{G_y\gamma'}{C}\,M\cos^2\varkappa,\\
U_z &= \frac{C}{m^3}\,\frac{G_z\gamma''}{C}\,M,
\end{aligned}$$

we shall have

$$\begin{aligned}
X' &= A\left[\frac{U_x}{G_x + a^2} + \frac{U_y}{G_y + a^2} + \frac{U_z}{G_z + a^2}\right],\\
Y' &= B\left[\frac{U_x}{G_x + b^2} + \frac{U_y}{G_y + b^2} + \frac{U_z}{G_z + b^2}\right],\\
Z' &= C\left[\frac{U_x}{G_x} + \frac{U_y}{G_y} + \frac{U_z}{G_z}\right].
\end{aligned}$$

By substituting in U_x, U_y, U_z, the last values we have given for γ, γ', γ'', and bearing in mind that

$$G_z - G_x = m^2, \quad G_y - G_x = m^2 \sin^2\theta, \quad G_z - G_y = m^2 \cos^2\theta,$$

the expressions for the first mentioned quantities become

$$\begin{aligned} U_x &= \frac{A+ae}{m^5}\sqrt{G_x(G_x+a^2)(G_x+b^2)}\,\frac{M\sin^2\varkappa}{\sin\theta}, \\ U_y &= \frac{B}{m^5}\sqrt{-G_y(G_y+a^2)(G_y+b^2)}\,\frac{M\cos^2\varkappa}{\sin\theta\cos\theta}, \\ U_z &= \frac{C}{m^5}\sqrt{G_z(G_z+a^2)(G_z+b^2)}\,\frac{M}{\cos\theta}, \end{aligned}$$

where, if γ, γ', γ'' have been taken positively, the three radicals must receive the sign of C.

Let it be required to find the component of this attracting force directed towards the centre of the ellipse described by the attracting planet. Putting $\mathrm{r}^2 = A^2 + B^2 + C^2$, we ought to multiply the components given above severally by the factors $\frac{A}{\mathrm{r}}$, $\frac{B}{\mathrm{r}}$, $\frac{C}{\mathrm{r}}$, and take the sum. Which, if we do, and have regard to relations, G_x, G_y, G_z satisfy as being the roots of the cubic in G; calling this component Δ, we have

$$\mathrm{r}\,\Delta = U_x + U_y + U_z.$$

But the component R_0 directed towards the Sun will be more important. Putting r_0 for the radius vector of the attracted planet, we have

$$r_0^2 = (A+ae)^2 + B^2 + C^2.$$

Then the components X, Y, Z ought to be multiplied severally by $\frac{A+ae}{r_0}$, $\frac{B}{r_0}$, $\frac{C}{r_0}$, and thus is obtained

$$r_0 R_0 = U_x + U_y + U_z + aeX'.$$

In fact, by multiplying X', Y', Z' severally by the three systems of three multipliers

$$\begin{array}{ccc} \dfrac{A+ae}{r_0}, & \dfrac{B}{r_0}, & \dfrac{C}{r_0}, \\ -\dfrac{B}{\sqrt{r_0^2-C^2}}, & \dfrac{A+ae}{\sqrt{r_0^2-C^2}}, & 0, \\ -\dfrac{(A+ae)\,C}{r_0\sqrt{r_0^2-C^2}}, & -\dfrac{BC}{r_0\sqrt{r_0^2-C^2}}, & \dfrac{\sqrt{r_0^2-C^2}}{r_0}, \end{array}$$

we shall arrive at a set of components, of which the first, already given, is directed toward the Sun, and the second is perpendicular thereto and lying in the plane of the attracting planet's orbit, while the third is perpendicular to both the preceding. If we call these new components X'', Y'', Z'', and if the angle between the planes of the orbits of the two planets be denoted by I, we shall have

$$R_0 = X'', \quad S_0 = Y'' \cos I - Z'' \sin I, \quad W_0 = Y'' \sin I + Z'' \cos I,$$

where R_0 is the component towards the Sun, S_0 the component perpendicular thereto and lying in the plane of the attracted planet's orbit, and W_0 the component perpendicular to that plane. But, if we make use of a rectangular set of elements instead of a polar, it is more commodious to refer the components to the plane and line of nodes of the attracted planet on the attracting planet's orbit. Let this ascending node be distant an arc $= \omega$ from the perihelion of the latter. Then the desired components X''', Y''', Z''' will be

$$\begin{aligned} X''' &= X' \cos \omega - Y' \sin \omega, \\ Y''' &= X' \cos I \sin \omega + Y' \cos I \cos \omega - Z' \sin I, \\ Z''' &= X' \sin I \sin \omega + Y' \sin I \cos \omega + Z' \cos I. \end{aligned}$$

It is often interesting to know the position of the great circles forming the principal axes of the sphero-conic in reference to the great circle marked out in the heavens by the plane of the attracting planet's orbit. If we call ☊ the longitude of the ascending node of one of the planes of symmetry of the cone on the plane xy, and i the inclination (always between 0° and 180°), and τ the angular distance of the centre of the sphero-conic from the node measured in the direction of increasing longitudes, the four following equations can be used for the determination of these quantities:

$$\tan i \sin \text{☊} = -\frac{A}{C}\,\frac{G_z}{G_z + a^2}, \qquad \tan i \cos \text{☊} = \frac{B}{C}\,\frac{G_z}{G_z + b^2},$$

$$\sin i \sin \tau = -\left[1 + \frac{A^2}{C^2}\left(\frac{G_x}{G_x + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_x}{G_z + b^2}\right)^2\right]^{-\frac{1}{2}},$$

$$\sin i \cos \tau = -\left[1 + \frac{A^2}{C^2}\left(\frac{G_y}{G_y + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_y}{G_y + b^2}\right)^2\right]^{-\frac{1}{2}},$$

where the signs of the two radicals may be taken positive or negative, thus corresponding to the four intersections of the two great circles with the one great circle.

To find the two semi-axes of the sphero-conic as measured by the arcs they subtend in the heavens, take the equation of the cone

$$\frac{\mathrm{X}^2}{G_x} + \frac{\mathrm{Y}^2}{G_y} + \frac{\mathrm{Z}^2}{G_z} = 0\,;$$

the section of the cone by the plane $\mathrm{Z} = \sqrt{G_z}$ gives the ellipse whose equation is

$$-\frac{\mathrm{X}^2}{G_x} - \frac{\mathrm{Y}^2}{G_y} = 1.$$

The semi-axis of this in the direction of the axis of X is $\sqrt{-G_x}$ and in the direction of the axis of Y, $\sqrt{-G_y}$. The greatest latitude η_0 of the planet moving on the sphero-conic and the greatest longitude λ_0 will be given by the equations

$$\tan \eta_0 = \sqrt{-\frac{G_y}{G_z}}, \quad \tan \lambda_0 = \sqrt{-\frac{G_x}{G_z}}\,.$$

The general equation connecting the variables η and λ will be obtained if, in the equation of the cone, we make

$$\mathrm{X} = \rho \cos \eta \sin \lambda, \quad \mathrm{Y} = \rho \sin \eta, \quad \mathrm{Z} = \rho \cos \eta \cos \lambda,$$

and thus is

$$\frac{\sin^2 \lambda}{G_x} + \frac{\tan^2 \eta}{G_y} + \frac{\cos^2 \lambda}{G_z} = 0.$$

These variables η and λ are expressed in terms of the perspective anomaly T as follows:

$$\cos \eta \cos \lambda = \varepsilon \sqrt{G_z}, \quad \cos \theta \sin \lambda = \varepsilon \sqrt{-G_x} \cos T, \quad \sin \eta = \varepsilon \sqrt{-G_y} \sin T.$$

We see that when $T = 0$, also $\eta = 0$ and $\tan \lambda = \sqrt{-\frac{G_x}{G_z}}$; and when $T = \frac{\pi}{2}$, $\tan \eta = \sqrt{-\frac{G_y}{G_z}}$, and $\lambda = 0$; when $T = \pi$, $\eta = 0$, $\tan \lambda = -\sqrt{\frac{G_x}{G_z}}$.

On the Solution of the Cubic in G and the Argument to be Employed in Tabulating the Elliptic Integrals.

A few words may be added in reference to the solution of the cubic in G,

$$\frac{A^2}{G + a^2} + \frac{B^2}{G + b^2} + \frac{C^2}{G} = 1.$$

Using r^2 for $A^2 + B^2 + C^2$, this equation expanded is

$$G^3 - [\mathrm{r}^2 - (a^2 + b^2)]\, G^2 + [a^2A^2 + b^2B^2 - (a^2 + b^2)\,\mathrm{r}^2 + a^2b^2]\, G - a^2b^2C^2 = 0.$$

It will be noticed that in obtaining the values of k, the modulus of the elliptic integrals involved, and of m, we do not need all the roots of this equation, but only their differences. Hence, it will be advantageous to put

$$G = J + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)], \quad \mathrm{a} = a^2 + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)],$$
$$\mathrm{b} = b^2 + \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)], \quad \mathrm{c} = \tfrac{1}{3}[\mathrm{r}^2 - (a^2 + b^2)]$$

The cubic will then take the form

$$\frac{A^2}{J + \mathrm{a}} + \frac{B^2}{J + \mathrm{b}} + \frac{C^2}{J + \mathrm{c}} = 1,$$

where J will serve us equally well as G; but here we have

$$\mathrm{a} + \mathrm{b} + \mathrm{c} = A^2 + B^2 + C^2,$$

and the developed equation in J will be

$$J^3 + [\mathrm{a}A^2 + \mathrm{b}B^2 + \mathrm{c}C^2 - \mathrm{a}^2 - \mathrm{b}^2 - \mathrm{c}^2 - \mathrm{ab} - \mathrm{bc} - \mathrm{ca}]\, J$$
$$- [\mathrm{bc}A^2 + \mathrm{ca}B^2 + \mathrm{ab}C^2 - \mathrm{abc}] = 0,$$

By the elimination of c and C and the partial reintroduction of a^2 and b^2, the shorter form is obtained,

$$J^3 - [\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]\, J + \mathrm{a}\,(b^2B^2 - \mathrm{ab}) + \mathrm{b}\,(a^2A^2 - \mathrm{ab}) = 0.$$

Employing the well-known trigonometric process for the solution of the cubic having all its roots real, we derive ψ (between the limits ± 90°) from

$$\sin 3\psi = \frac{\sqrt{27}}{2}\,\frac{\mathrm{a}\,(b^2B^2 - \mathrm{ab}) + \mathrm{b}\,(a^2A^2 - \mathrm{ab})}{[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{3}{2}}}.$$

The roots of the cubic are then

$$J_x = -\frac{2}{\sqrt{3}}[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin(60^\circ + \psi),$$

$$J_y = \frac{2}{\sqrt{3}}[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin\psi,$$

$$J_z = \frac{2}{\sqrt{3}}[\mathrm{a}^2 + \mathrm{ab} + \mathrm{b}^2 - a^2A^2 - b^2B^2]^{\frac{1}{2}} \sin(60^\circ - \psi).$$

To render these equations simpler, we will put

$$\mathrm{a} + \mathrm{b} = 2n \cos \nu, \quad \sqrt{3}\,(\mathrm{a} - \mathrm{b}) = 2n \sin \nu, \quad a^2A^2 + b^2B^2 - 2\,\mathrm{ab} = n^2\mu \cos \xi,$$

$$\frac{1}{\sqrt{3}}(a^2A^2 - b^2B^2) = n^2\mu \sin \xi.$$

With these modifications we have

$$\sin 3\psi = \frac{\sqrt{27}}{2}\ \frac{\mu \cos (\nu + \xi)}{(1 - \mu \cos \xi)^{\frac{3}{2}}},$$

$$J_x = -\frac{2}{\sqrt{3}}\, n\, (1 - \mu \cos \xi)^{\frac{1}{2}} \sin (60^\circ + \psi),$$

$$J_y = \frac{2}{\sqrt{3}}\, n\, (1 - \mu \cos \xi)^{\frac{1}{2}} \sin \psi,$$

$$J_z = \frac{2}{\sqrt{3}}\, n\, (1 - \mu \cos \xi \sin)^{\frac{1}{2}} (60 - \psi).$$

The modulus of the elliptic integrals involved, k, will be given by the equation

$$k^2 = \sin^2 \theta = \frac{J_y - J_x}{J_z - J_x} = \frac{\cos (60^\circ - \psi)}{\cos \psi.},$$

and the quantity we have designated by m, by the formula

$$m^2 = J_z - J_x = 2n\,(1 - \mu \cos \xi)\,\tfrac{1}{2} \cos \psi.$$

The two elliptic integrals $M \sin^2 x$, $M \cos^2 x$ are functions of ψ; consequently, they may be tabulated with the argument ψ or any function of ψ as, for instance, with $\sin 3\psi$, that is, with the absolute discriminant of the cubic. We have $\cos^2 \theta = \dfrac{\sin (30^0 - \psi)}{\cos \psi}$, and, if we put x^4 for the second member of this, we have

$$\left[\frac{\sin \frac{\theta}{2}}{1 + \sqrt{\cos \theta}}\right]^2 = \frac{1}{2}\,\frac{1 - x}{1 + x} = \frac{q + q^9 + q^{25} + \ldots.}{1 + 2\,(q^4 + q^{16} + q^{36} + \ldots.)}.$$

The value of the nome q can be derived from the infinite series (three terms suffice except in very unusual cases),

$$q = \frac{1}{2}\,\frac{1 - x}{1 + x} + \frac{1}{16}\left(\frac{1 - x}{1 + x}\right)^5 + \frac{15}{512}\left(\frac{1 - x}{1 + x}\right)^9 + \ldots.$$

As G_x, G_y, G_z enter into many of the preceding formulas, it may be useful to note that

$$\begin{array}{lll} G_x = J_x + \mathrm{c}, & G_y = J_y + \mathrm{c}, & G_z = J_z + \mathrm{c}, \\ G_x + a^2 = J_x + \mathrm{a}, & G_y + a^2 = J_y + \mathrm{a}, & G_z + a^2 = J_z + \mathrm{a}, \\ G_x + b^2 = J_x + \mathrm{b}, & G_y + b^2 = J_y + \mathrm{b}, & G_z + b^2 = J_z + \mathrm{b}. \end{array}$$

Variation of the Elements of a Planet through Perturbation.

The elements we select for use are defined thus: x, y, z, denoting the rectangular coordinates of a planet referred to the Sun as origin, adopt the elements c_x, c_y, c_z, f_x, f_y, f_z such that

$$c_x = \frac{ydz - zdy}{dt}, \qquad c_y = \frac{zdx - xdz}{dt}, \qquad c_z = \frac{xdy - ydx}{dt},$$

$$f_x = \frac{c_z dy - c_y dz}{dt} - \frac{\mu x}{r}, \quad f_y = \frac{c_x dz - c_z dx}{dt} - \frac{\mu y}{r}, \quad f_z = \frac{c_y dx - c_x dy}{dt} - \frac{\mu z}{r},$$

where we note that μ is the sum of the masses of the Sun and planet, and r is the radius vector of the latter. These six elements are not independent but satisfy the relation

$$c_x f_x + c_y f_y + c_z f_z = 0.$$

The additional element needed to complete the number six is the element everywhere attached by addition to the time. With these constants the two equations of the path of the planet in space are

$$c_x x + c_y y + c_z z = 0, \quad \mu r + f_x x + f_y y + f_x z = c_x^2 + c_y^2 + c_z^2 = k^2.$$

To understand the correlation of the different terms of these equations it must be borne in mind that μ is a constant of three dimensions in reference to the linear unit. Consequently, the c are of two dimensions and the f of three dimensions in reference to this unit.

The second equation belongs to a quadric surface not, in general, referred to its axes of symmetry. Removing the radical from it, it becomes

$$(\mu^2 - f_x^2)\, x^2 + (\mu^2 - f_y^2)\, y^2 + (\mu^2 - f_z^2)\, z^2 - 2 f_x f_y\, xy - 2 f_y f_z\, yz - 2 f_z f_x\, zx$$
$$+ 2k^2 f_x x + 2k^2 f_y y + 2k^2 f_z z - k^4 = 0.$$

The cubic, which must be solved in order that this may be referred to its axes of symmetry, is

$$\chi^3-[3\mu^2-(f_x^2+f_y^2+f_z^2)]\,\chi^2+[(\mu^2-f_x^2)(\mu^2-f_y^2)+(\mu^2-f_y^2)(\mu^2-f_z^2)$$
$$+(\mu^2-f_z^2)(\mu^2-f_x^2)-(f_x^2f_y^2+f_y^2f_z^2+f_z^2f_x^2)]\,\chi-\mu^4\,[\mu^2-(f_x^2+f_y^2+f_z^2)]=0\,.$$

But, if e denotes the eccentricity, we have $f_x^2+f_y^2+f_z^2=\mu^2e^2$; thus the preceding equation reduces to

$$\chi^3-\mu^2\,(3-e^2)\,\chi^2+\mu^4\,(3-2e^2)\,\chi-\mu^6\,(1-e^2)=0\,.$$

Put $\chi=\mu^2\varkappa$, then

$$\varkappa^3-(3-e^2)\,\varkappa^2+(3-2e^2)\,\varkappa-(1-e^2)=0\,,$$

or

$$(\varkappa-1)^3+e^2\,(\varkappa-1)^2=0\,.$$

The three roots are $\varkappa=1$, $\varkappa=1$, $\varkappa=1-e^2$. The second equation of the orbit of the planet is, therefore, a quadric of revolution about the major axis, in the present investigation an ellipsoid, as we suppose $e^2<1$. The expression $c_x^2+c_y^2+c_z^2$ is invariant, as also is $f_x^2+f_y^2+f_z^2$, both being independent of the orientation of the axes of coordinates. The system of principal axes will be arrived at if we suppose $f_y=0\,,f_z=0$, which imply also $c_x=0$, as is shown by the relation $c_xf_x+c_y\,f_y+c_zf_z=0$, unless we have $f_x=0$, when, the quadric being a sphere, all systems of axes are principal. The equations of the orbit then take the form

$$c_yy+c_zz=0,\quad \mu r+\mu ex=k^2,$$

where $k^2=c_y^2+c_z^2$. The radical removed, the second equation becomes

$$\mu^2\,(1-e^2)\,x^2+\mu^2y^2+\mu^2z^2+2k^2\mu ex-k^4=0\,.$$

But, a being the semi-axis major, $k^2=\mu a\,(1-e^2)$, therefore, the preceding equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{a^2\,(1-e^2)}+2\,\frac{e}{a}\,x-(1-e^2)=0,$$

or

$$\frac{(x+ae)^2}{a^2}+\frac{y^2+z^2}{a^2\,(1-e^2)}=1\,.$$

The adoption of the attracted planet instead of the centre of the Sun as the origin of coordinates renders it necessary, while employing the elements of the planet, to substitute $-x_0$, $-y_0$, $-z_0$ for x, y, z. Thus, with this notation, the equations of the Sun's path in space are

$$c_xx_0+c_yy_0+c_zz_0=0,\quad \mu r_0-f_xx_0-f_yy_0-f_zz_0=k^2.$$

The values of the differentials with respect to the time of the constants c and f must now be stated. Denoting the mass of the attracting planet by m', the perturbative function R for secular perturbations is $\frac{m'}{\rho}$, and we have

$$\begin{aligned}
\frac{dc_x}{dt} &= y_0\frac{\partial R}{\partial z} - z_0\frac{\partial R}{\partial y} = \frac{m'}{\rho^3}(z_0 y - y_0 z),\\
\frac{dc_y}{dt} &= z_0\frac{\partial R}{\partial x} - x_0\frac{\partial R}{\partial z} = \frac{m'}{\rho^3}(x_0 z - z_0 x),\\
\frac{dc_z}{dt} &= x_0\frac{\partial R}{\partial y} - y_0\frac{\partial R}{\partial x} = \frac{m'}{\rho^3}(y_0 x - x_0 y),\\
\frac{df_x}{dt} &= \frac{dz_0}{dt}\frac{dc_y}{dt} - \frac{dy_0}{dt}\frac{dc_z}{dt} + \frac{m'}{\rho^3}(c_z y - c_y z),\\
\frac{df_y}{dt} &= \frac{dx_0}{dt}\frac{dc_z}{dt} - \frac{dz_0}{dt}\frac{dc_x}{dt} + \frac{m'}{\rho^3}(c_x z - c_z x),\\
\frac{df_z}{dt} &= \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt} + \frac{m'}{\rho^3}(c_y x - c_x y).\ *
\end{aligned}$$

There is still one element to be added to the preceding to complete the number of six independent constants; this is the mean longitude at epoch. The well-known equation for its variation shows that a portion is immediately derivable from the motion of the perihelion, and another from that of the node; what remains (calling the element l) is given by the equation

$$\frac{dl}{dt} = -2\frac{an}{\mu}\left[x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right] = 2an\frac{m'}{\mu}\left[x_0\frac{x}{\rho^3} + y_0\frac{y}{\rho^3} + z_0\frac{z}{\rho^3}\right],$$

where n denotes the mean motion of the attracted planet.

The system just given is the most general as respects the orientation of the axes of coordinates. Let us now specialize by taking, for the plane of xy, the plane of the orbit of the attracted planet. This makes $c_x = 0$, $c_y = 0$, $c_z = k$ and the equation between the c and the f shows that, in consequence, we have $f_z = 0$. Then our equations become

$$\begin{aligned}
\frac{dc_x}{dt} &= -\frac{m'}{\rho^3}y_0 z, & \frac{df_x}{dt} &= -\frac{dy_0}{dt}\frac{dk}{dt} + k\frac{m'}{\rho^3}y,\\
\frac{dc_y}{dt} &= \frac{m'}{\rho^3}x_0 z, & \frac{df_y}{dt} &= \frac{dx_0}{dt}\frac{dk}{dt} - k\frac{m'}{\rho^3}x,\\
\frac{dk}{dt} &= \frac{m'}{\rho^3}(y_0 x - x_0 y), & \frac{df_z}{dt} &= \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt}.
\end{aligned}$$

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* For these formulas consult Laplace, Mécanique Celeste, Tom. I, Liv. II, Art. 64.

But our specialization, applied to the definitions of the f, gives

$$-k\frac{dx_0}{dt} = \mu\frac{y_0}{r_0} - f_y, \qquad k\frac{dy_0}{dt} = \mu\frac{x_0}{r_0} - f_x.$$

These values enable us to eliminate the differentials of x_0, y_0 from the preceding equations. By differentiating the equation between the c and f we get

$$f_x\frac{dc_x}{dt} + f_y\frac{dc_y}{dt} + k\frac{df_z}{dt} = 0.$$

The substitution of preceding values in this renders it an identity, hence the equation for $\frac{df_z}{dt}$ is superfluous. Moreover, after use in substitution, the equation for $\frac{dk}{dt}$ is no longer needed, since the secular motion of the semi-axis a vanishes. Thus the equations to be employed are reduced to the five following:

$$\begin{aligned}
\frac{dc_x}{dt} &= -m'y_0\frac{z}{\rho^3},\\
\frac{dc_y}{dt} &= m'x_0\frac{z}{\rho^3},\\
\frac{df_x}{dt} &= \frac{m'}{k}\left(f_x - \mu\frac{x_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) + m'k\frac{y}{\rho^3},\\
\frac{df_y}{dt} &= \frac{m'}{k}\left(f_y - \mu\frac{y_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) - m'k\frac{x}{\rho^3},\\
\frac{dl}{dt} &= 2an\frac{m'}{\mu}\left(x_0\frac{x}{\rho^3} + y_0\frac{y}{\rho^3}\right).
\end{aligned}$$

If χ denote the longitude of the perihelion of the attracted planet, we have $f_x = \mu e\cos\chi$, $f_y = \mu e\sin\chi$; also, if i is the inclination and ☊ the longitude of the ascending node, $c_x = k\sin i\sin\text{☊}$, $c_y = -k\sin i\cos\text{☊}$; whence

$$\begin{aligned}
\frac{d(\sin i\sin\text{☊})}{dt} &= -\frac{m'}{k}y_0\frac{z}{\rho^3},\\
\frac{d(\sin i\cos\text{☊})}{dt} &= -\frac{m'}{k}x_0\frac{z}{\rho^3},\\
\frac{d(e\sin\chi)}{dt} &= \frac{m'}{k}\left(e\sin\chi - \frac{y_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) - \frac{m'}{\mu}k\frac{x}{\rho^3},\\
\frac{d(e\cos\chi)}{dt} &= \frac{m'}{k}\left(e\cos\chi - \frac{x_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) + \frac{m'}{\mu}k\frac{y}{\rho^3}.
\end{aligned}$$

The point of departure for longitudes is still undetermined; but, if this is taken to be the ascending node of the attracted planet's orbit on that of the attracting planet, we can employ the X''', Y''', Z''' we have previously defined. Thus, using brackets with subscript 0 to denote integration along the orbit of the attracting planet we have the equations

$$\begin{aligned}
\left[\frac{d\,(\sin i \sin \Omega)}{dt}\right]_0 &= -\frac{m'}{k}\, y_0\, Z''', \\
\left[\frac{d\,(\sin i \cos \Omega)}{dt}\right]_0 &= -\frac{m'}{k}\, x_0\, Z''', \\
\left[\frac{d\,(e \sin \chi)}{dt}\right]_0 &= \frac{m'}{k}\left(e \sin \chi - \frac{y_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) - \frac{m'}{\mu}\, k\, X''', \\
\left[\frac{d\,(e \cos \chi)}{dt}\right]_0 &= \frac{m'}{k}\left(e \cos \chi - \frac{x_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) + \frac{m'}{\mu}\, k\, Y''', \\
\left[\frac{dl}{dt}\right]_0 &= 2an\,\frac{m'}{\mu}\left(x_0 X''' + y_0 Y'''\right).
\end{aligned}$$

The integration round the orbit of the disturbed planet has still to be executed in order to arrive at the secular motion of the elements. For this we are confined to the use of mechanical quadratures. Here we may use either of the three anomalies, or any variable which will show the position of the planet on its orbit, as the independent variable.

WEST NYACK, *March* 13, 1901.